Published for SISSA by Springer
Received: October 22, 2013
Accepted: December 11, 2013
Published: January 16, 2014

Discrete θ and the 5d superconformal index

Oren Bergman,[a] **Diego Rodríguez-Gómez**[b] **and Gabi Zafrir**[a]

[a] *Department of Physics, Technion, Israel Institute of Technology, Haifa, 32000, Israel*
[b] *Department of Physics, Universidad de Oviedo, Avda. Calvo Sotelo 18, 33007, Oviedo, Spain*

E-mail: bergman@physics.technion.ac.il, d.rodriguez.gomez@uniovi.es, gabizaf@techunix.technion.ac.il

Abstract: 5d Yang-Mills theory with an Sp(N) gauge group admits a discrete analog of the θ parameter. We describe the origin of this parameter in $\mathcal{N} = 1$ theories from Type I' string theory, and study its effect on the 5d superconformal fixed point theories with an Sp(1) = SU(2) gauge group by computing the superconformal index. Our result confirms the lack of global symmetry enhancement in the so-called $\tilde{E}_1$ theory.

Keywords: Brane Dynamics in Gauge Theories, Supersymmetric gauge theory, Solitons Monopoles and Instantons

ArXiv ePrint: 1310.2150

Article funded by SCOAP[3].

doi:10.1007/JHEP01(2014)079

Contents

1 Introduction

Interacting quantum field theories in 5d are non-renormalizable and therefore do not generically exist as microscopic theories. Nevertheless there is compelling evidence that there exist strongly-interacting $\mathcal{N} = 1$ supersymmetric fixed point theories in 5d, some of which have relevant deformations corresponding to ordinary gauge theories with matter [1–5].

The simplest set of examples has a gauge group SU(2) and $N_f \leq 7$ fundamental hypermultiplets [1]. This set of fixed point theories is particularly interesting since it was argued to exhibit an exotic global symmetry E_{N_f+1}. This is not visible in the gauge theory action, which exhibits only an $SO(2N_f) \times U(1)_T$ global symmetry, where $U(1)_T$ is the topological symmetry associated to the conserved current $j_T = *\mathrm{Tr}(F \wedge F)$. It can however be inferred by a particular string theory embedding of the gauge theory using a D4-brane in Type I' string theory. When the D4-brane coincides with the O8-plane and N_f D8-branes at one of the boundaries, the low energy supersymmetric gauge theory has an $Sp(1) = SU(2)$ gauge symmetry, and N_f matter multiplets in the fundamental representation. The fixed point theory corresponds to the limit where the dilaton blows up locally at this boundary, which is possible only for $N_f \leq 7$. In particular, this explains the enhancement of the global symmetry to E_{N_f+1}, as a result of the enhancement of the 9d gauge symmetry on the D8-branes due to massless D0-branes [6–8].

The enhanced global symmetry has recently been confirmed by an impressive calculation of the superconformal index, including instanton corrections, for the SU(2) theory with $N_f \leq 5$ [9]. The index exhibits explicitly the conserved current multiplets associated with the E_{N_f+1} symmetry. The extra currents not contained in $SO(2N_f)$ correspond to instanton-particles.

For $N_f = 0$ it was argued in [2] that there exists a second fixed point theory, the so-called $\tilde{E}_1$ theory, in which the $\mathrm{U}(1)_T$ global symmetry is not enhanced. It was argued that this theory is associated with a discrete analog of the θ parameter in 5d [3].

In this note we will revisit the $\tilde{E}_1$ theory. We will explain how to embed it in the Type I' description by a previously overlooked discrete choice in the background. We will then compute the superconformal index for this theory by adapting the computation of [9] to this discrete choice, and exhibit the lack of symmetry enhancement. We will also offer an alternative computation of the index, treating SU(2) as SU(N) with $N = 2$. In this approach the $\tilde{E}_1$ theory corresponds to the theory with a CS term at level $\kappa = 1$. Along the way we also clarify how the SU(2) theory with CS level $\kappa = 2$ is equivalent to that with $\kappa = 0$. We conclude by mentioning some future directions and raising some questions about the string theory interpretation.

2 The $\tilde{E}_1$ theory

In [2] Morrison and Seiberg showed that there are two more interacting fixed points that can be reached by relevant deformations of the E_{N_f+1} theories. Starting with the E_2 theory, there is a two-parameter space of relevant deformations spanned by the bare coupling $t_0 \equiv 1/g_0^2$ and the flavor mass m. These can be thought of as VEVs of scalars in background vector multiplets associated to the global $E_2 = \mathrm{SU}(2) \times \mathrm{U}(1)$ symmetry. In particular m is associated to the U(1) part, and the combination $m_0 \equiv t_0 - 2m$ to the SU(2) part. The origin of the (m_0, m) plane is the E_2 theory. Turning on $m > 0$ with $m_0 = 0$, one flows to the E_1 theory with $E_1 = \mathrm{SU}(2)$ global symmetry. On the other hand for $m_0 > 0$ and $m = 0$ one flows to the the free D_1 theory with $\mathrm{SO}(2) \times \mathrm{U}(1)$ global symmetry. For $m_0 > 0$ and $m < 0$ there is a singular locus where $m_0 + 4m = 0$, along which the effective coupling diverges. This suggests that in this direction one reaches a new interacting fixed point with only a U(1) global symmetry, the so-called $\tilde{E}_1$ theory. This theory has only one relevant parameter $s = m_0 + 4m$. For $s > 0$ it flows back to the free D_1 theory, but for $s < 0$ it flows to another interacting fixed point without any global symmetry, the E_0 theory.

A similar conclusion was reached by considering the geometric realization of the E_{N_f+1} fixed points as singular CY spaces with collapsed del-Pezzo surfaces [2, 3]. The flows described above correspond to shrinking a 2-cycle inside the del-Pezzo surface and then blowing one up in the CY space transverse to the del-Pezzo surface, i.e. a flop transition. For the surface describing the E_2 theory, B_2, there are two inequivalent choices leading either to the Hirzebruch surface $B_1 = F_1$ or to the direct product $CP^1 \times CP^1$. The latter corresponds to the E_1 theory and the former to the $\tilde{E}_1$ theory.

Related to this description is the description of the 5d fixed point theories in terms of Type IIB 5-brane webs [5]. In this picture 5d gauge theories are constructed using a configuration of 5-branes in Type IIB string theory, in which the gauge fields live on the worldvolumes of D5-branes that are suspended between NS5-branes. The web-like configuration is supported by semi-infinite (p, q)5-branes carrying the appropriate charges for charge conservation. The webs for the E_1 and $\tilde{E}_1$ theories are shown in figure 1. Flavors can be added by attaching semi-infinite D5-branes. In particular with one flavor, it is easy

JHEP01(2014)079

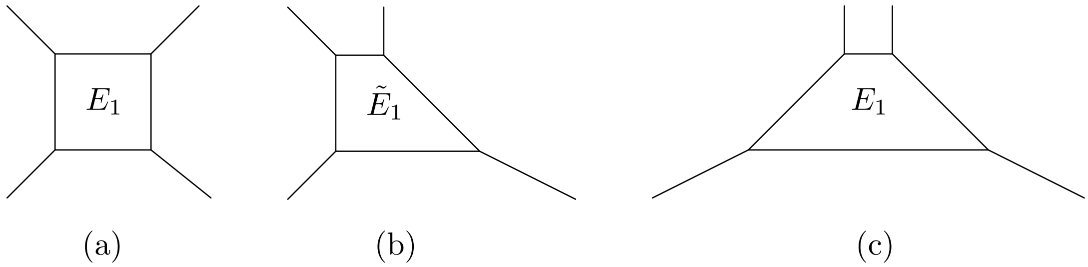


Figure 1. Pure SU(2): (a) E_1 theory, (b) $\tilde{E}_1$ theory, (c) another representation of E_1. The webs are drawn with the assumption that $g_s = 1$ and $C_0 = 0$.

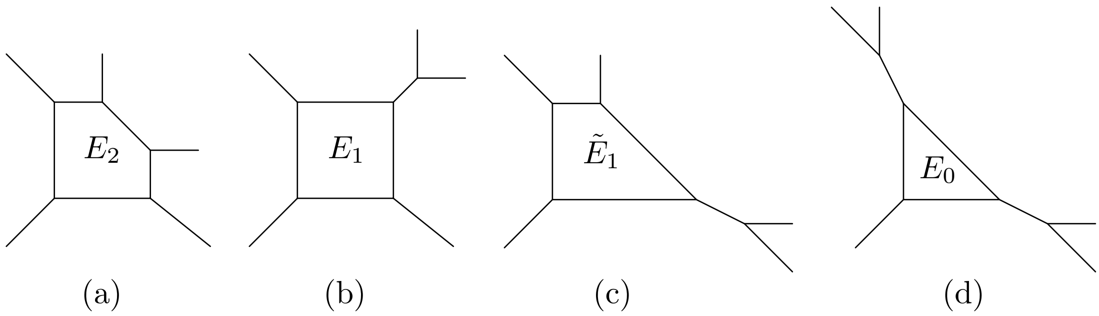


Figure 2. SU(2) with one flavor: (a) E_2 theory, (b) positive mass deformation, (c) negative mass deformation, (d) E_0 theory.

to generate the flow from the E_2 theory to either the E_1 or $\tilde{E}_1$ theory, figure 2(a,b,c). The E_0 theory can be reached by deforming the $\tilde{E}_1$ web as in figure 2(d).

Since the E_1 and $\tilde{E}_1$ theories are obtained by deforming with opposite signs for the flavor mass m, it was argued that the difference between the two gauge theories is a 5d analog of the θ parameter of Yang-Mills theory [3]. Indeed, 5d SU(2) gauge theory, and more generally Sp(N) gauge theory, admits a $\mathbb{Z}_2$-valued θ parameter associated with $\pi_4(\mathrm{SU}(2)) = \mathbb{Z}_2$. This is a discrete 5d analog of the familiar 4d θ parameter associated with $\pi_3(\mathrm{SU}(2)) = \mathbb{Z}$. In 4d, Euclidean gauge field configurations with instanton number $n \in \pi_3(\mathrm{SU}(2))$ are weighted by a phase $e^{in\theta}$. The non-trivial element of $\pi_4(\mathrm{SU}(2))$ is associated to a $\mathbb{Z}_2$-charged instanton in 5d.[1] The 5d path integral thus has two contributions, and we have a choice of taking the sum or difference of the two. This is interpreted as the discrete choice of the θ parameter. When massive flavors are present, this parameter can be absorbed into the sign of their mass. Therefore, if there is a massless flavor, θ is physically irrelevant. There is only one theory with one flavor, the E_2 fixed point. But in deforming it by giving mass to the flavor there are two choices, leading to the E_1 and $\tilde{E}_1$ theories.

[1]This is related to the 4d global anomaly of [10]. This anomaly was observed by constructing a non-trivial path in configuration space that interpolates between a 4d gauge field configuration and its global transformation by an SU(2) element in the non-trivial class of $\pi_4(\mathrm{SU}(2))$. Along such a path an odd number of eigenvalues of the Dirac operator change sign, so there is an anomaly if the are an odd number of fermions in the fundamental representation. This path can be thought of as a $\mathbb{Z}_2$-valued instanton in 5d.

3 Type I' string description

One can now ask how the discrete θ parameter of the 5d $\mathrm{Sp}(N)$ gauge theory is realized in the Type I' description. This should correspond to some discrete choice that one can make in the Type I' background. Indeed such a choice exists. In 10d Type IIA string theory there is a one-form gauge field in the RR sector, C_1. In reducing to 9d by compactifying on a circle this gives rise to a 9d θ parameter, $\theta_9 = \int_{S^1} C_1$. Type I' string theory involves a projection by the combination of spatial and worldsheet reflections, under which θ_9 is odd, and therefore projected out. However, since $\theta_9 \sim \theta_9 + 2\pi$, there is a discrete remnant corresponding to the choice of $\theta_9 = 0$ or π.[2]

Associated to θ_9 is a non-BPS D(-1)-brane [12] whose contribution to the path integral comes with a phase $e^{i\theta_9}$. (The D(-1)-brane is related by T-duality to the non-BPS D0-brane of Type I string theory [13, 14].) The D(-1)-brane can be regarded as a discrete "gauge instanton" in the 9d $O(2N_f)$ theory on the D8-branes. Its charge is associated with the non-trivial element of $\pi_8(O) = \mathbb{Z}_2$. Alternatively, it can also be regarded as an instanton in the 5d $\mathrm{Sp}(N)$ theory on the D4-branes, or as an instanton in the 1d $O(k)$ theory on a collection of k D0-branes. Mathematically, this is the statement of Bott periodicity, which relates

$$\pi_8(O) = \pi_4(\mathrm{Sp}) = \pi_0(O)\,. \tag{3.1}$$

The latter is simply a Wilson line in Euclidean time, where the $O(k)$ gauge field undergoes a gauge transformation in the negative-determinant component of $O(k)$. (This is related by T-duality to the realization of the Type I D0-brane as a Wilson line in the Type I D1-brane [15].) We are therefore lead to identify the θ parameters in the different dimensions:

$$\theta_9 = \theta_5 = \theta_1\,. \tag{3.2}$$

It is useful to construct a background in which θ_9 changes between its two possible values. This is achieved by considering the 9d "magnetic" dual of the non-BPS D(-1)-brane, which is a non-BPS D7-brane. Depending on the size of the interval in the Type I' background, this corresponds to either a D7-brane localized on the O8-plane, or to a D8-brane-anti-D8-brane combination stretched along the interval, figure 3(a) [12]. (These are T-dual to the D8-brane and D7-brane of Type I string theory [14].) Either way, this creates a domain wall in 9d across which θ_9 changes from 0 to π. So when the D(-1)-brane crosses the D7-brane it acquires a -1 phase. This also means that the 5d θ parameter jumps across the wall. In the absence of flavors, i.e., no D8-branes transverse to the interval, this creates a clear separation between two regions in 5d with different values of θ, corresponding to the two distinct vacua of the 5d $\mathrm{Sp}(N)$ gauge theory. When a flavor D8-brane is added the configuration becomes unstable; there is a tachyon at the intersection of the D8-brane and the D7-brane, leading to the absorption of the D7-brane by the D8-brane. In the stretched D8-$\overline{\mathrm{D8}}$ description of the non-BPS D7-brane the branes break and reconnect, as shown in figure 3(c). In other words, the domain wall disappears, and the two vacua are physically equivalent. This is exactly what we expect from the point of view of the 5d gauge theory. The θ parameter can be transformed away in the presence of massless fermion matter.

[2] A similar choice exists in Type I string theory, implying the existence of a new 10d string [11].

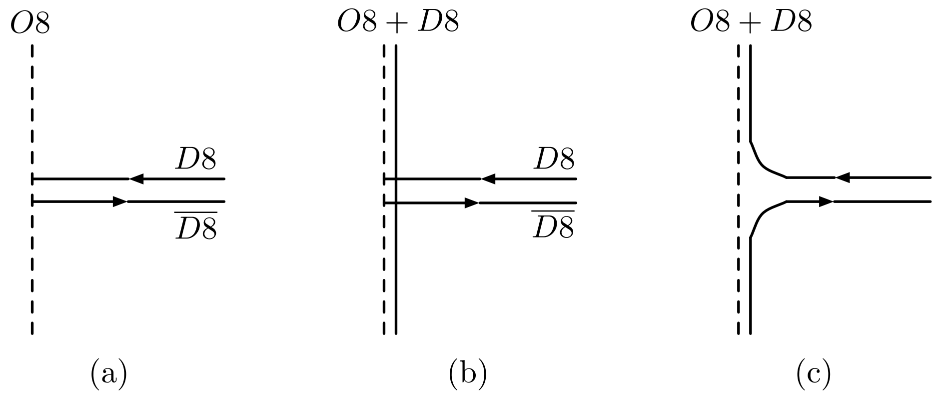


Figure 3. Type I' description of the θ parameter: (a) A transverse D8-$\overline{\text{D8}}$ forms a θ domain wall, (b) adding a flavor D8-brane leads to an instability, (c) the branes break and reconnect, eliminating the domain wall.

4 Superconformal index

The superconformal index is a characteristic of superconformal field theories that essentially counts a class of BPS operators [16]. As such, it is protected from continuous deformations of the theory, and can be used as a diagnostic of non-perturbative physics like duality and enhanced symmetries. Given a supercharge Q and its conjugate in radial quantization S, the associated superconformal index is defined in general by

$$I(\mu_i) = \mathrm{Tr}\left[(-1)^F e^{-\beta\Delta} e^{\mu_i q_i}\right] , \tag{4.1}$$

where $\Delta = \{Q, S\}$ is the Hamiltonian in radial quantization, μ_i are chemical potentials associated to symmetries that commute with Q, and q_i are the corresponding charges. The BPS states contributing non-trivially to I satisfy $\Delta = 0$. The above expression can be translated to a Euclidean functional integral in the field theory on $S^{d-1} \times S^1$ with appropriately twisted periodicity conditions on S^1.

The computation is greatly simplified using supersymmteric localization, which reduces the functional integral to ordinary matrix integrals. This was done for 5d $\mathcal{N} = 1$ theories in [9], leading to an expression that factorizes into a 1-loop determinant contribution and a contribution of "instanton particles":[3]

$$I(x, y, m_i, q) = \int [\mathcal{D}\alpha]\, I_{\text{loop}}(\alpha, x, y, m_i)\, |I_{\text{inst}}(\alpha, x, y, m_i, q)|^2 . \tag{4.2}$$

The integral is taken over the holonomy matrix α and includes the Haar measure of the gauge group. The other parameters are fugacities associated with the Cartan generators of the global symmetry $\mathrm{SO}(5) \times \mathrm{SU}(2)_R \times \mathrm{SO}(2N_f) \times \mathrm{U}(1)_T$.

[3]We will refer to these as instanton particles to differentiate them from the gauge instantons discussed previously. The latter are true instantons in the sense of being pointlike in 5d, whereas the former are particles with world-lines in 5d.

For each type of superfield, the 1-loop contribution is given by a Plethystic exponential of the one-letter index

$$I_{\text{loop}} = PE[f(\cdot)] = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f(\cdot^n)\right] . \tag{4.3}$$

The vector multiplet contributes

$$f_{\text{vector}}(x, y, \alpha) = -\frac{x(y + \frac{1}{y})}{(1 - xy)(1 - \frac{x}{y})} \sum_{\mathbf{r} \in \mathbf{R}} e^{-i\mathbf{r}\cdot\alpha} , \tag{4.4}$$

where $\mathbf{R}$ is the root lattice, and each matter multiplet contributes

$$f_{\text{matter}}(x, y, m_i, \alpha) = \frac{x}{(1 - xy)(1 - \frac{x}{y})} \sum_{\mathbf{w} \in \mathbf{W}} \sum_{i=1}^{N_f} \left(e^{i\mathbf{w}\cdot\alpha + im_i} + e^{-i\mathbf{w}\cdot\alpha - im_i}\right) , \tag{4.5}$$

where the sum is over the appropriate weights in the weight lattice.

The instanton contribution is given by a product of a contribution from instanton particles located at the south pole of the S^4 and a contribution of anti-instanton particles located at the north pole. Each is expressed as a power series in the instanton number,

$$I_{\text{inst}}(\alpha, x, y, m_i, q) = 1 + qZ_1(\alpha, x, y, m_i) + q^2 Z_2(\alpha, x, y, m_i) + \cdots , \tag{4.6}$$

where Z_k is the 5d Nekrasov k-instanton partition function [17]. The computation of Z_k involves following the ADHM procedure for quantizing the multi-instanton moduli space, described by supersymmetric gauge quantum mechanics with a dual gauge group. This boils down to a contour integral over the Cartan subalgebra of the dual gauge group. The result depends on both the gauge group and the matter content. Exact results for SU(N), Sp(N) and SO(N) with various matter were obtained in [18].

4.1 Pure SU(2)

For SU(2) = Sp(1) the dual group for k instantons is $O(k)$. This is of course the gauge symmetry of the D0-brane theory in the Type I' string theory description. This group has two disconnected components denoted $O(k)_+ = \text{SO}(k)$ and $O(k)_-$. The latter is the set of determinant -1 elements of $O(k)$ and does not form a group. For $k = 2n$, the torus action of the group is generated by $\text{diag}(e^{i\sigma_2\phi_1}, \dots, e^{i\sigma_2\phi_n})$ for $O(2n)_+$ and by $\text{diag}(e^{i\sigma_2\phi_1}, \dots, e^{i\sigma_2\phi_{n-1}}, \sigma_3)$ for $O(2n)_-$. In other words the dimension of the Cartan subalgebra of $O(2n)_-$ is smaller by 1 than that of $O(2n)_+$. For $k = 2n + 1$ the torus action is generated by $\text{diag}(e^{i\sigma_2\phi_1}, \dots, e^{i\sigma_2\phi_n}, \pm 1)$. The k-instanton index has two contributions coming from the the two components $O(k)_+$ and $O(k)_-$, corresponding, respectively, to the sectors of the supersymmetric quantum mechanics without and with the discrete Wilson line.

The general expressions for the two contributions, Z_k^+ and Z_k^-, were given in [9]. Let us reproduce here just the 1-instanton functions. The $O(1)_+$ part contributes

$$Z_1^+ = \frac{x^2}{(1 - xy)\left(1 - \frac{x}{y}\right)(x - s)\left(x - \frac{1}{s}\right)} , \tag{4.7}$$

where s is the gauge fugacity $e^{i\alpha}$, and the $O(1)_-$ part contributes

$$Z_1^- = \frac{x^2}{(1-xy)\left(1-\frac{x}{y}\right)(x+s)\left(x+\frac{1}{s}\right)} . \tag{4.8}$$

The latter is equivalent to inserting a parity twist $(-1)^P$ in the trace defining the index, where P is the element $-1 \in O(1)$. The only parity odd modes come from the gauge moduli, hence the sign flip in front of s. The full instanton index is given by combining the two parts, or equivalently by parity-projecting the spectrum. But here one has a choice of projecting by either $\frac{1}{2}(1+(-1)^P)$ or $\frac{1}{2}(1-(-1)^P)$. This is precisely the choice of the θ parameter, which gives a phase $e^{i\theta}$ in the contribution with the Wilson line. The even-parity projection gives

$$Z_1^{\theta=0} = \frac{1}{2}\left(Z_1^+ + Z_1^-\right) = \frac{x^2(1+x^2)}{(1-xy)\left(1-\frac{x}{y}\right)(1-(xs)^2)\left(1-\left(\frac{x}{s}\right)^2\right)} , \tag{4.9}$$

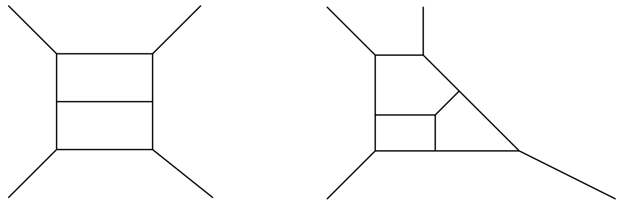
which is the result of [9] for pure SU(2). For $\theta = \pi$ we take the difference of (4.7) and (4.8), which gives

$$Z_1^{\theta=\pi} = \pm\frac{1}{2}\left(Z_1^+ - Z_1^-\right) = \pm\frac{x^3\left(s+\frac{1}{s}\right)}{(1-xy)\left(1-\frac{x}{y}\right)(1-(xs)^2)\left(1-\left(\frac{x}{s}\right)^2\right)} . \tag{4.10}$$

We have included a possible sign ambiguity, due to a potential shift in the fermion number relative to the $\theta = 0$ case. We are not able to fix the sign from first principles, but we will give an indirect argument that the minus sign is the correct one.

It is illuminating to expand the result in powers of x, since this orders the contributions roughly according to their scaling dimension. The leading order contribution corresponds to the ground state of the instanton particle. We see that to leading order, $Z_1^{\theta=0} \sim x^2$, whereas $Z_1^{\theta=\pi} \sim x^3\left(s+\frac{1}{s}\right)$. This shows that the ground state of the instanton particle is charged under the gauge symmetry in the $\tilde{E}_1$ theory, and is gauge-neutral in the E_1 theory. This is something we can also see from the description of BPS states as string-webs inside the 5-brane-web [19], figure 4. On the Coulomb branch of the E_1 theory, the instanton particle is described by a D-string between the two NS5-branes. It is therefore uncharged with respect to the SU(2) gauge symmetry at the origin. (Although it becomes charged under the unbroken U(1) on the Coulomb branch due to the one-loop CS term). In the $\tilde{E}_1$ theory on the other hand, the instanton particle is described a 3-string web with a fundamental string prong ending on one of the D5-branes. This corresponds to an SU(2) charge in the fundamental representation, which is precisely what we see in the leading term in the index.

Let us now return to the question of the overall sign in (4.10). This is related to the question of whether the ground state is bosonic or fermionic, and as we will now argue, the ground state of the instanton particle in the $\tilde{E}_1$ theory is most likely fermionic. The basic idea is to examine the symmetry properties of the 2-instanton operator, and to compare it with the symmetry properties of the product of two 1-instanton operators. For free



(a) (b)

Figure 4. The instanton particle: (a) in the E_1 theory, (b) in the $\tilde{E}_1$ theory.

particles, the two particle state should be a symmetrized product if they are bosons and an antisymmetrized product if they are fermions. This should be seen by comparing the 2-instanton partition function with the symmetrized or antisymmetrized product of the 1-instanton partition function. The instantons are not free particles, so we will not get an equality. Nevertheless, the leading terms in x are dominated by the symmetrized (or antisymmetrized) product of the 1-instanton moduli space, and so we should be able to determine which possibility of the two is more likely. The symmetry properties of the products are encoded in the Plethystic exponential. Extracting the 2-instanton term, in the bosonic case we would get

$$PE\left[\frac{qx^3\left(s+\frac{1}{s}\right)}{(1-xy)\left(1-\frac{x}{y}\right)(1-(xs)^2)\left(1-\left(\frac{x}{s}\right)^2\right)}\right]\Bigg|_{q^2} \approx \left(1+s^2+\frac{1}{s^2}\right)x^6+O(x^7)\,, \tag{4.11}$$

whereas in the fermionic case we would get

$$PE\left[-\frac{qx^3\left(s+\frac{1}{s}\right)}{(1-xy)\left(1-\frac{x}{y}\right)(1-(xs)^2)\left(1-\left(\frac{x}{s}\right)^2\right)}\right]\Bigg|_{q^2} \approx x^6+O(x^7)\,. \tag{4.12}$$

Comparing with the expansion of the 2-instanton index,

$$Z_2^{\theta=\pi} \approx x^6+O(x^7)\,, \tag{4.13}$$

we conclude that the ground state of the instanton in the $\tilde{E}_1$ theory is actually fermionic, and therefore that the correct sign in (4.10) is the minus sign.

We can now compute the superconformal index using (4.2). One can deduce from [9] that for small x, $Z_k^{\pm} \sim \mathcal{O}(x^{2k})$. So to compute the index to $\mathcal{O}(x^{2k})$, we generically need to include instanton contributions up instanton number k. For the $\theta = 0$ theory the authors of [9] found[4]

$$
\begin{aligned}
I^{\theta=0} =\ & 1+\chi_{\mathbf{3}}(q)x^2+\chi_{\mathbf{2}}(y^2)[1+\chi_{\mathbf{3}}(q)]x^3+\Big(\chi_{\mathbf{3}}(y^2)\big[1+\chi_{\mathbf{3}}(q)\big]+1+\chi_{\mathbf{5}}(q)\Big)x^4 \\
&+\Big(\chi_{\mathbf{4}}(y^2)\big[1+\chi_{\mathbf{3}}(q)\big]+\chi_{\mathbf{2}}(y^2)\big[1+\chi_{\mathbf{3}}(q)+\chi_{\mathbf{5}}(q)\big]\Big)x^5 \\
&+\Big(\chi_{\mathbf{5}}(y^2)\big[1+\chi_{\mathbf{3}}(q)\big]+\chi_{\mathbf{3}}(y^2)\big[1+\chi_{\mathbf{3}}(q)+\chi_{\mathbf{5}}(q)+\chi_{\mathbf{3}}^2(q)\big]+\chi_{\mathbf{3}}(q)+\chi_{\mathbf{7}}(q)-1\Big)x^6 \\
&+\Big(\chi_{\mathbf{6}}(y^2)\big[1+\chi_{\mathbf{3}}(q)\big]+\chi_{\mathbf{4}}(y^2)\big[2+4\chi_{\mathbf{3}}(q)+2\chi_{\mathbf{5}}(q)\big] \\
&\qquad\qquad+\chi_{\mathbf{2}}(y^2)\big[1+3\chi_{\mathbf{3}}(q)+2\chi_{\mathbf{5}}(q)+\chi_{\mathbf{7}}(q)\big]\Big)x^7 \\
&+\Big(\chi_{\mathbf{7}}(y^2)\big[1+\chi_{\mathbf{3}}(q)\big]+\chi_{\mathbf{5}}(y^2)\big[3\chi_{\mathbf{5}}(q)+5\chi_{\mathbf{3}}(q)+4\big] \\
&+\chi_{\mathbf{3}}(y^2)\big[2\chi_{\mathbf{7}}(q)+3\chi_{\mathbf{5}}(q)+7\chi_{\mathbf{3}}(q)+2\big]+\chi_{\mathbf{9}}(q)+2\chi_{\mathbf{5}}(q)+2\chi_{\mathbf{3}}(q)+3\Big)x^8 \\
&+\mathcal{O}(x^9)\,.
\end{aligned}
\tag{4.14}
$$

This exhibits manifestly the enhancement of the global $\mathrm{U}(1)_T$ symmetry to $E_1 = \mathrm{SU}(2)$. In particular the conserved current multiplets contribute to the coefficient of x^2, which is $\chi_{\mathbf{3}}(q) = 1 + q + 1/q$. The three contributions come respectively from the perturbative $\mathrm{U}(1)_T$ current and two charged currents corresponding to an instanton particle (D0-brane) and an anti-instanton particle (anti-D0-brane). For the $\theta = \pi$ theory we find

$$
\begin{aligned}
I^{\theta=\pi} =\ & 1+x^2+2\chi_{\mathbf{2}}(y^2)x^3+\Big(1+2\chi_{\mathbf{3}}(y^2)\Big)x^4+\Big(2\chi_{\mathbf{4}}(y^2)+\chi_{\mathbf{2}}(y^2)\Big)x^5 \\
&+\Big(2\chi_{\mathbf{5}}(y^2)+3\chi_{\mathbf{3}}(y^2)+q^2+\frac{1}{q^2}\Big)x^6 \\
&+\left(2\chi_{\mathbf{6}}(y^2)+4\chi_{\mathbf{4}}(y^2)+4\chi_{\mathbf{2}}(y^2)+\chi_{\mathbf{2}}(y^2)\left(q^2+\frac{1}{q^2}\right)\right)x^7 \\
&+\left(2\chi_{\mathbf{7}}(y^2)+7\chi_{\mathbf{5}}(y^2)+7\chi_{\mathbf{3}}(y^2)+4+\chi_{\mathbf{3}}(y^2)\left(q^2+\frac{1}{q^2}\right)+q^3+\frac{1}{q^3}\right)x^8 \\
&+O(x^9)\,.
\end{aligned}
\tag{4.15}
$$

As anticipated, there is no symmetry enhancement in this theory. In fact, it is clear from (4.10) that the instanton will only start to contribute at $\mathcal{O}(x^6)$ in this case. Our result also matches perfectly the computation of the index of the $\tilde{E}_1$ theory done in [20], to the order it was done there.

4.2 Adding flavor

Flavor hypermultiplets contribute fermionic zero modes that give factors of the form $(e^{im_i/2} \mp e^{-im_i/2})$ in the numerator of the instanton index [9], where m_i are the chemical potentials associated to the flavor symmetries (these can be thought of as the masses of the flavor hypermultiplets). The sign is correlated with the sign of $O(k)_\pm$. For $O(k)_+$ the sign is negative due to the $(-1)^F$ operation, and for $O(k)_-$, corresponding to the insertion of $(-1)^P$, the sign is positive since the flavors are parity-odd. For one flavor, the

[4] The result is expressed in terms of SU(2) characters. For example $\chi_{\mathbf{2}}(x) = \sqrt{x} + \frac{1}{\sqrt{x}}$, $\chi_{\mathbf{3}}(x) = x + 1 + \frac{1}{x}$.

JHEP01(2014)079

1-instanton functions are given by

$$Z_1^{\pm} = \frac{x^2(e^{im/2} \mp e^{-im/2})}{(1-xy)\left(1-\frac{x}{y}\right)(x \mp s)\left(x \mp \frac{1}{s}\right)} . \tag{4.16}$$

For the two choices of θ we then get:

$$Z_1^{\theta=0} = \frac{x^2\left(e^{im/2}(1+x^2) - xe^{-im/2}\left(s+\frac{1}{s}\right)\right)}{(1-xy)\left(1-\frac{x}{y}\right)(1-(xs)^2)\left(1-\left(\frac{x}{s}\right)^2\right)} , \tag{4.17}$$

$$Z_1^{\theta=\pi} = \frac{x^2\left(e^{-im/2}(1+x^2) - xe^{im/2}\left(s+\frac{1}{s}\right)\right)}{(1-xy)\left(1-\frac{x}{y}\right)(1-(xs)^2)\left(1-\left(\frac{x}{s}\right)^2\right)} , \tag{4.18}$$

where in the second case we have used the overall minus sign of (4.10). The two indices are related by changing the sign of m, and they are equal for $m=0$. This is as expected, since the θ parameter becomes unphysical in the presence of massless fermions in the fundamental representation. This also reinforces our argument for the sign in (4.10).

4.3 An alternative approach

An alternative way to compute the SU(2) instanton index is to treat the gauge group as SU(N) with $N=2$. Practically speaking, the SU(N) Nekrasov partition function is really for U(N), but we can freeze-out the overall U(1) by setting its fugacity to 1. However the distinction is important when we include a CS term. Although there is no possible CS term for SU(2), one is possible for U(2). We will see that from the SU(2) point of view this reduces to the discrete θ parameter. The Nekrasov 1-instanton partition function for SU(2) with CS level κ is given by [9]:

$$Z_1^{\mathrm{SU}(2)_\kappa} = \frac{1}{2\pi i} \oint \frac{u^{1+\kappa}(1-x^2)du}{(1-xy)\left(1-\frac{x}{y}\right)(u-xs)\left(u-\frac{x}{s}\right)\left(u-\frac{s}{x}\right)\left(u-\frac{1}{xs}\right)} , \tag{4.19}$$

where s is the SU(2) gauge fugacity as before. The integral is taken on the unit circle in the u-plane. There are four poles, but if we take $x \ll 1$ only the two at $u = xs$ and $u = \frac{x}{s}$ contribute, and the result is

$$Z_1^{\mathrm{SU}(2)_\kappa} = \frac{x^{2+\kappa}\left[(s^{2-\kappa}-s^{\kappa})x^2 + s^{2+\kappa} - s^{-\kappa}\right]}{(1-xy)\left(1-\frac{x}{y}\right)(1-(xs)^2)\left(1-\left(\frac{x}{s}\right)^2\right)(s^2-1)} . \tag{4.20}$$

At CS level $\kappa = 0$ this gives

$$Z_1^{\mathrm{SU}(2)_0} = \frac{x^2(1+x^2)}{(1-xy)\left(1-\frac{x}{y}\right)(1-(xs)^2)\left(1-\left(\frac{x}{s}\right)^2\right)} , \tag{4.21}$$

in agreement with the result for the E_1 theory, namely the SU(2) theory with $\theta = 0$ (4.9). At CS level $\kappa = 1$ we get

$$Z_1^{\mathrm{SU}(2)_1} = \frac{x^3\left(s+\frac{1}{s}\right)}{(1-xy)\left(1-\frac{x}{y}\right)(1-(xs)^2)\left(1-\left(\frac{x}{s}\right)^2\right)} , \tag{4.22}$$

in agreement, up to a sign, with the $\tilde{E}_1$ theory (4.10).

JHEP01(2014)079

The sign difference above suggests that the procedure of freezing-out the overall U(1) by setting its fugacity to 1 leaves a residue of the form $(-1)^\kappa$. More generally, the result for Z_k with fundamental matter in the SU(N) formalism seems to have a U(1) residue $(-1)^{k\left(\kappa+N_f/2\right)}$. We suspect that this is associated with the mixed CS term inherent in the decomposition of U(2) to U(1) $\times$ SU(2),

$$S_{\text{mixed CS}} \propto \kappa \int \hat{A} \wedge \text{Tr}(F \wedge F)\,. \tag{4.23}$$

The $N_f/2$ contribution reflects the one-loop shift of the CS level due to the flavors. It would be interesting to understand this better.

It is also interesting to consider the theory with CS level $\kappa = 2$. The result for the 1-instanton partition function is given by

$$Z_1^{\text{SU}(2)_2} = \frac{x^4\left[s^2+1+\frac{1}{s^2}-x^2\right]}{(1-xy)\left(1-\frac{x}{y}\right)(1-(xs)^2)\left(1-\left(\frac{x}{s}\right)^2\right)}\,. \tag{4.24}$$

However this result is problematic since it is not invariant under $x \to \frac{1}{x}$, unlike the results for $\kappa = 0$ and $\kappa = 1$. This transformation is part of the superconformal group; it's an element of $\text{SU}(2)_x \subset \text{SO}(5)$. Therefore it should be respected by the instanton index. The above result for the index is missing states that are required in order to form complete representations of the superconformal group. We claim that the missing sector can be accounted for by adding a term Δ to (4.24), where

$$\Delta = \frac{x^2}{(1-xy)\left(1-\frac{x}{y}\right)}\,. \tag{4.25}$$

The sum is

$$Z_1^{\text{SU}(2)_2} + \Delta = \frac{x^2(1+x^2)}{(1-xy)\left(1-\frac{x}{y}\right)(1-(xs)^2)\left(1-\left(\frac{x}{s}\right)^2\right)}\,, \tag{4.26}$$

and is invariant under $x \to \frac{1}{x}$. Indeed it is precisely $Z_1^{\text{SU}(2)_0}$, namely the 1-instanton index of the E_1 theory. This is what we expect. The CS level 2 theory is really the SU(2) theory with $\theta = 2\pi \sim 0$. It corresponds to the second 5-brane web realization of the E_1 theory shown in figure 1(c). More generally, we claim that the full index, including all instanton corrections, should be multiplied by $PE[q\Delta]$. A more detailed derivation and interpretation of this result will appear elsewhere [21]. As a teaser, let us however mention that the same term must be added more generally for $\text{SU}(N)_N$, showing that the fixed point theory corresponding to $\text{SU}(N)_N$ has an enhanced SU(2) global summetry.

5 Conclusions

By properly implementing the effect of the 5d θ parameter on the instanton particle in the 5d $\mathcal{N} = 1$ SU(2) theory, we have computed the superconformal index of the $\tilde{E}_1$ theory. Our result confirms the lack of global symmetry enhancement, and agrees with other approaches.

The generalization to $\mathrm{Sp}(N)$ with an antisymmetric hypermultiplet is straightforward, and can be obtained in an analogous manner from the results of [9]. The $\theta = \pi$ theory will not have an enhanced global symmetry. In fact, the instanton contribution starts at a power of x that scales with N, so that at large N the index of the $\tilde{E}_1$ theory becomes purely perturbative (for a computation of the large N perturbative index see [22]).

It would be interesting to explore further the Type I' string theory description of the discrete θ parameter, and in particular its effect on the D0-brane. Can the difference in the ground states of the instanton particle in the two theories be understood in terms of some stringy mechanism?

It would also be interesting to understand the supergravity dual of the large N $\tilde{E}_1$ theory, and related quiver theories, extending the results of [23, 24].

Acknowledgments

O.B. would like to thank the High Energy Physics group at the University of Oviedo for their hospitality. O.B. is supported in part by the Israel Science Foundation under grants no. 392/09, and 352/13, the US-Israel Binational Science Foundation under grants no. 2008-072, and 2012-041, the German-Israeli Foundation for Scientific Research and Development under grant no. 1156-124.7/2011, and by the Technion V.P.R Fund. G.Z. is supported in part by Israel Science Foundation under grant no. 392/09. D.R-G is supported by the Ramón y Cajal fellowship RyC-2011-07593, as well as by the Spansih Ministry of Science and Education grant FPA2012-35043-C02-02.

References

[1] N. Seiberg, *Five-dimensional SUSY field theories, nontrivial fixed points and string dynamics*, *Phys. Lett.* **B 388** (1996) 753 [hep-th/9608111] [INSPIRE].

[2] D.R. Morrison and N. Seiberg, *Extremal transitions and five-dimensional supersymmetric field theories*, *Nucl. Phys.* **B 483** (1997) 229 [hep-th/9609070] [INSPIRE].

[3] M.R. Douglas, S.H. Katz and C. Vafa, *Small instantons, Del Pezzo surfaces and type-I-prime theory*, *Nucl. Phys.* **B 497** (1997) 155 [hep-th/9609071] [INSPIRE].

[4] K.A. Intriligator, D.R. Morrison and N. Seiberg, *Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces*, *Nucl. Phys.* **B 497** (1997) 56 [hep-th/9702198] [INSPIRE].

[5] O. Aharony and A. Hanany, *Branes, superpotentials and superconformal fixed points*, *Nucl. Phys.* **B 504** (1997) 239 [hep-th/9704170] [INSPIRE].

[6] J. Polchinski and E. Witten, *Evidence for heterotic - type-I string duality*, *Nucl. Phys.* **B 460** (1996) 525 [hep-th/9510169] [INSPIRE].

[7] D. Matalliotakis, H.-P. Nilles and S. Theisen, *Matching the BPS spectra of heterotic Type I and Type I-prime strings*, *Phys. Lett.* **B 421** (1998) 169 [hep-th/9710247] [INSPIRE].

JHEP01(2014)079

- [8] O. Bergman, M.R. Gaberdiel and G. Lifschytz, *String creation and heterotic type-I' duality*, *Nucl. Phys.* **B 524** (1998) 524 [hep-th/9711098] [INSPIRE].
- [9] H.-C. Kim, S.-S. Kim and K. Lee, *5-dim Superconformal Index with Enhanced En Global Symmetry*, *JHEP* **10** (2012) 142 [arXiv:1206.6781] [INSPIRE].
- [10] E. Witten, *An* SU(2) *Anomaly*, *Phys. Lett.* **B 117** (1982) 324 [INSPIRE].
- [11] S. Sethi, *A New String in Ten Dimensions?*, *JHEP* **09** (2013) 149 [arXiv:1304.1551] [INSPIRE].
- [12] O. Bergman, E.G. Gimon and P. Hořava, *Brane transfer operations and T duality of nonBPS states*, *JHEP* **04** (1999) 010 [hep-th/9902160] [INSPIRE].
- [13] A. Sen, *Type I D particle and its interactions*, *JHEP* **10** (1998) 021 [hep-th/9809111] [INSPIRE].
- [14] E. Witten, *D-branes and k-theory*, *JHEP* **12** (1998) 019 [hep-th/9810188] [INSPIRE].
- [15] A. Sen, SO(32) *spinors of type-I and other solitons on brane - anti-brane pair*, *JHEP* **09** (1998) 023 [hep-th/9808141] [INSPIRE].
- [16] J. Kinney, J.M. Maldacena, S. Minwalla and S. Raju, *An Index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209 [hep-th/0510251] [INSPIRE].
- [17] N.A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [hep-th/0206161] [INSPIRE].
- [18] N. Nekrasov and S. Shadchin, *ABCD of instantons*, *Commun. Math. Phys.* **252** (2004) 359 [hep-th/0404225] [INSPIRE].
- [19] O. Aharony, A. Hanany and B. Kol, *Webs of (p,q) five-branes, five-dimensional field theories and grid diagrams*, *JHEP* **01** (1998) 002 [hep-th/9710116] [INSPIRE].
- [20] A. Iqbal and C. Vafa, *BPS Degeneracies and Superconformal Index in Diverse Dimensions*, arXiv:1210.3605 [INSPIRE].
- [21] O. Bergman, D. Rodriguez-Gomez and G. Zafrir, *5-Brane Webs, Symmetry Enhancement, and Duality in 5d Supersymmetric Gauge Theory*, to appear.
- [22] O. Bergman, D. Rodriguez-Gomez and G. Zafrir, *5d superconformal indices at large-N and holography*, *JHEP* **08** (2013) 081 [arXiv:1305.6870] [INSPIRE].
- [23] A. Brandhuber and Y. Oz, *The D-4 - D-8 brane system and five-dimensional fixed points*, *Phys. Lett.* **B 460** (1999) 307 [hep-th/9905148] [INSPIRE].
- [24] O. Bergman and D. Rodriguez-Gomez, *5d quivers and their AdS_6 duals*, *JHEP* **07** (2012) 171 [arXiv:1206.3503] [INSPIRE].